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Choreographic three bodies on the lemniscate

Toshiaki Fujiwara[1], **Hiroshi Fukuda**[2] **and Hiroshi Ozaki**[3]

[1] Faculty of General Studies, Kitasato University, Kitasato 1-15-1, Sagamihara, Kanagawa 228-8555, Japan
[2] School of Administration and Informatics, University of Shizuoka, 52-1 Yada, Shizuoka 422-8526, Japan
[3] Department of Physics, Tokai University, 1117 Kitakaname, Hiratsuka, Kanagawa 259-1292, Japan

E-mail: fujiwara@clas.kitasato-u.ac.jp, fukuda@u-shizuoka-ken.ac.jp and ozaki@keyaki.cc.u-tokai.ac.jp

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Abstract
We show that choreographic three bodies $\{\mathbf{x}(t), \mathbf{x}(t+T/3), \mathbf{x}(t-T/3)\}$ of period T on the lemniscate, $\mathbf{x}(t) = (\hat{\mathbf{x}} + \hat{\mathbf{y}}\,\mathrm{cn}(t))\mathrm{sn}(t)/(1+\mathrm{cn}^2(t))$ parametrized by the Jacobian elliptic functions sn and cn with modulus $k^2 = (2+\sqrt{3})/4$, conserve the centre of mass and the angular momentum, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the orthogonal unit vectors defining the plane of the motion. They also conserve the moment of inertia, the kinetic energy, the sum of squares of the curvature, the product of distances and the sum of squares of distances between bodies. We find that they satisfy the equation of motion under the potential energy $\sum_{i<j}\left((1/2)\ln r_{ij} - (\sqrt{3}/24)r_{ij}^2\right)$ or $\sum_{i<j}(1/2)\ln r_{ij} - \sum_i(\sqrt{3}/8)r_i^2$, where r_{ij} is the distance between bodies i and j, and r_i the distance from the origin. The first term of the potential energies is the universal gravitation in two dimensions but the second term is a mutual repulsive force or a repulsive force from the origin, respectively. Then, geometric construction methods for the positions of the choreographic three bodies are given.

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1. Introduction

Choreographic motion of N bodies is a periodic motion on a closed orbit, N bodies chase each other on this orbit with equal time-spacing. Recently, choreographic motions under Newtonian gravity have been found and paid attention to. Moore [1] found a figure-eight three-body choreographic solution by numerical calculations. Chenciner and Montgomery [2] gave a rigorous proof of the existence of a choreographic figure-eight three-body solution. At the same time, Simó [3, 4] found many remarkable choreographic N-body solutions by numerical calculations. Although the exact form of these choreographic N-body solutions is still unknown, Simó's figure-eight choreographic three-body solution is very similar to an affine transformed lemniscate [4]. Here Simó's figure-eight solution means the E orbit among his four figure-eight orbits in [4].

Concerning the relation between the lemniscate and Newton's equation of motion, it is well known [5] that a point particle on the lemniscate $r^2 = \cos(2\theta)$ with $\theta = 1/2 \sin^{-1}(2t)$, $-1 \leqslant 2t \leqslant 1$ satisfies the equation of motion under the central potential $U(r) = -1/(2r^6)$. Here r, θ and t represent the radius, the azimuthal angle and the time, respectively. This motion is not periodic. This particle starts from the origin at $t = -1/2$, travels the right leaf of the lemniscate, and finally collides with the origin at $t = 1/2$. The two-body problem on the lemniscate is derived from the one-body problem. Two particles of equal masses start from the origin at $t = -1/2$ in opposite directions, each particle travels the left leaf or right leaf of the half-size lemniscate, and collides with each other at the origin at $t = 1/2$. No analytic solution is known for more than three bodies on the lemniscate.

Being stimulated by Moore, Chenciner, Montgomery and Simó's remarkable study, we investigated the physical and geometrical properties of the choreographic motion of three bodies on the lemniscate. We describe the results in section 2 and proof is given in section 3. In section 4, we discuss the relation between the choreographic three bodies on the lemniscate and the rectangular hyperbola, and give a geometrical method to determine the positions of the choreographic three bodies on the lemniscate. Section 5 gives a summary. Sections 2–4 describe the properties of choreographic three-body motion on the lemniscate. Some properties common to Moore, Chenciner, Montgomery and Simó's figure-eight will be summarized in section 5.

2. Choreographic three bodies on the lemniscate

Let us parametrize by using the Jacobian elliptic functions sn and cn the lemniscate $\mathbf{x}(t) = x(t)\hat{\mathbf{x}} + y(t)\hat{\mathbf{y}}$ which satisfies $(x^2 + y^2)^2 = x^2 - y^2$ as follows:

$$\begin{cases} x(t) = \dfrac{\mathrm{sn}(t)}{1 + \mathrm{cn}^2(t)} \\ y(t) = \dfrac{\mathrm{sn}(t)\mathrm{cn}(t)}{1 + \mathrm{cn}^2(t)} \end{cases} \tag{1}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the two basic orthogonal unit vectors defining the plane of the motion. This is a smooth periodic motion on the lemniscate with period

$$T = 4K \tag{2}$$

where

$$K = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

is the complete elliptic integral of the first kind of the modulus k in the definitions of the Jacobian elliptic functions

$$\mathrm{sn}^{-1}(t) = \int_0^t \frac{\mathrm{d}x}{\sqrt{(1 - x^2)(1 - k^2x^2)}}$$

and

$$\mathrm{cn}(t) = (1 - \mathrm{sn}^2(t))^{1/2}.$$

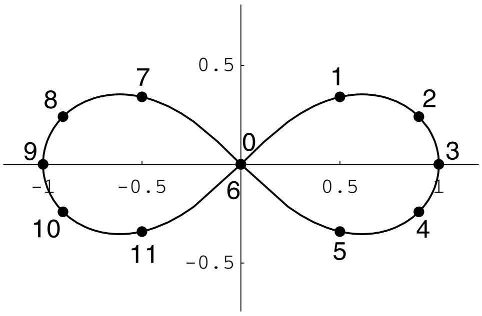

Figure 1. The lemniscate and the positions of $\mathbf{x}(t)$ with modulus $k^2 = (2+\sqrt{3})/4$ at $t = jK/3$ for $j = 0, 1, 2, \ldots, 11$. Full circles ● labelled by j represent the positions of $\mathbf{x}(t)$.

The positions of the choreographic three bodies are

$$\{\mathbf{x}_1(t), \mathbf{x}_2(t), \mathbf{x}_3(t)\} = \{\mathbf{x}(t), \mathbf{x}(t+4K/3), \mathbf{x}(t-4K/3)\}. \tag{3}$$

In the following, we use notation $\mathbf{v}(t) = \mathrm{d}\mathbf{x}(t)/\mathrm{d}t$ and $\mathbf{a}(t) = \mathrm{d}^2\mathbf{x}(t)/\mathrm{d}t^2$.

Straight calculation shows a relation between the curvature

$$\rho^{-1}(t) = \frac{|\mathbf{v}(t) \times \mathbf{a}(t)|}{|\mathbf{v}(t)|^3}$$

and the distance from the origin

$$\rho^{-2}(t) = 9\mathbf{x}^2(t) \tag{4}$$

which is, of course, the parametrization invariant relation. Also, we get a relation between the velocity and the distance from the origin,

$$\mathbf{v}^2(t) + \left(k^2 - \tfrac{1}{2}\right)\mathbf{x}^2(t) = \tfrac{1}{2} \tag{5}$$

for arbitrary modulus k.

Conservation of the centre of mass

$$\mathbf{x}(t) + \mathbf{x}(t+4K/3) + \mathbf{x}(t-4K/3) = \mathbf{0} \tag{6}$$

is satisfied if and only if the value of the modulus is

$$k^2 = \frac{2+\sqrt{3}}{4} \tag{7}$$

as shown in section 3. Parametrization in (1), (3) and (7) defines the motion of the choreographic three bodies on the lemniscate. In figure 1, the lemniscate and the positions $\mathbf{x}(t)$ at $t = jK/3$ for $j = 0, 1, 2, \ldots, 11$ are shown. The order of magnitude of the label j represents the direction of the motion, and the points having the same label in modulo 4 are the positions of the choreographic three bodies (3) on the lemniscate at $t = 0, K/3, 2K/3, K, \ldots$.

We find that this motion conserves the moment of inertia and angular momentum

$$\sum \mathbf{x}_i^2 = \sqrt{3} \tag{8}$$

$$\sum \mathbf{x}_i \times \mathbf{v}_i = 0 \tag{9}$$

as shown in section 3. Conservation of the moment of inertia and relations (4), (5) yield the conservation of the sum of squares of the curvature and the kinetic energy

$$\sum \rho_i^{-2} = 9\sqrt{3} \tag{10}$$

$$\sum \mathbf{v}_i^2 = \tfrac{3}{4}. \tag{11}$$

Simple algebra shows that (6) implies $\sum_{i<j}(\mathbf{x}_i - \mathbf{x}_j)^2 = 3\sum_i \mathbf{x}_i^2$. Thus, we get another conservation

$$\sum_{i<j}(\mathbf{x}_i - \mathbf{x}_j)^2 = 3\sqrt{3}. \tag{12}$$

The above conservation laws suggest that the choreographic three bodies on the lemniscate (1), (3) and (7) may satisfy an equation of motion under some interaction potential energy. We find that it surely satisfies an equation of motion under an attractive force proportional to the inverse of the distance with an extra repulsive force proportional to the distance

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathbf{x}(t) = \mathbf{F}_{\mathrm{attr}} + \mathbf{F}_{\mathrm{rep}} \\ &\mathbf{F}_{\mathrm{attr}} = \frac{1}{2}\left\{\frac{\mathbf{x}(t+4K/3) - \mathbf{x}(t)}{(\mathbf{x}(t+4K/3) - \mathbf{x}(t))^2} + \frac{\mathbf{x}(t-4K/3) - \mathbf{x}(t)}{(\mathbf{x}(t-4K/3) - \mathbf{x}(t))^2}\right\} \end{aligned} \tag{13}$$

as shown in section 3. The repulsive force $\mathbf{F}_{\mathrm{rep}}$ can be expressed in two ways:

$$\mathbf{F}_{\mathrm{rep1}} = \frac{\sqrt{3}}{4}\mathbf{x}(t) \tag{14}$$

or

$$\mathbf{F}_{\mathrm{rep2}} = -\frac{\sqrt{3}}{12}\left\{\left(\mathbf{x}\left(t + \frac{4K}{3}\right) - \mathbf{x}(t)\right) + \left(\mathbf{x}\left(t - \frac{4K}{3}\right) - \mathbf{x}(t)\right)\right\}. \tag{15}$$

Due to the conservation of the centre of mass, the two repulsive forces are equal, $\mathbf{F}_{\mathrm{rep1}} = \mathbf{F}_{\mathrm{rep2}}$, on the orbit. The corresponding potential energy for the equation of motion with $\mathbf{F}_{\mathrm{rep1}}$ is

$$U = \sum_{i<j}\frac{1}{2}\ln r_{ij} - \sum_i \frac{\sqrt{3}}{8}\mathbf{x}_i^2 \tag{16}$$

and with $\mathbf{F}_{\mathrm{rep2}}$ is

$$V = \sum_{i<j}\left\{\frac{1}{2}\ln r_{ij} - \frac{\sqrt{3}}{24}r_{ij}^2\right\} \tag{17}$$

where r_{ij} is the distance between points i and j, i.e., $r_{ij} = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)^2}$.

With the conservation of the kinetic energy and the moment of inertia, the conservation of total energy with potential energy U, in (16), implies the conservation of the product of r_{ij}^2. Evaluating the value at $t = 0$, we get

$$r_{12}^2(t)r_{23}^2(t)r_{31}^2(t) = \frac{3\sqrt{3}}{2}. \tag{18}$$

3. Proof

3.1. Conservation of the centre of mass

Since $\mathbf{x}(K) = \hat{\mathbf{x}}$, the $\hat{\mathbf{x}}$-component of the other two points must be $x(K \pm 4K/3) = -1/2$. This equation gives

$$\mathrm{sn}(K/3) = \sqrt{3} - 1. \tag{19}$$

Evaluating $\mathrm{sn}(10K/3)$ in two ways,

$$\mathrm{sn}(10K/3) = \mathrm{sn}(3K + K/3) = -\mathrm{cn}(K/3)/\mathrm{dn}(K/3)$$

Table 1. Values of $\mathrm{sn}(t)$, $\mathrm{cn}(t)$, $\mathrm{dn}(t)$.

t	$\mathrm{sn}(t)$	$\mathrm{cn}(t)$	$\mathrm{dn}(t)$
$K/3$	$\sqrt{3}-1$	$3^{1/4}(\sqrt{3}-1)/\sqrt{2}$	$1/\sqrt{2}$
$2K/3$	$3^{1/4}(\sqrt{3}-1)$	$2-\sqrt{3}$	$(\sqrt{3}-1)/2$
$4K/3$	$3^{1/4}(\sqrt{3}-1)$	$-2+\sqrt{3}$	$(\sqrt{3}-1)/2$
$5K/3$	$\sqrt{3}-1$	$-3^{1/4}(\sqrt{3}-1)/\sqrt{2}$	$1/\sqrt{2}$

and

$$\mathrm{sn}(10K/3) = \mathrm{sn}(4K - 2K/3) = -\frac{2\mathrm{sn}(K/3)\mathrm{cn}(K/3)\mathrm{dn}(K/3)}{(1 - k^2\mathrm{sn}^4(K/3))}$$

we get

$$k^2 = \frac{1 - 2\mathrm{sn}(K/3)}{\mathrm{sn}^4(K/3) - 2\mathrm{sn}^3(K/3)}. \tag{20}$$

Here

$$\mathrm{dn}(t) = (1 - k^2\mathrm{sn}^2(t))^{1/2}.$$

Substituting the value of (19) into (20) gives the value of k^2 in (7). Inversely, the value of the modulus in (7) gives the value of $\mathrm{sn}(K/3)$ in (19). The values of $\mathrm{sn}(t), \mathrm{cn}(t), \mathrm{dn}(t)$ at some t are shown in table 1. To complete the proof of (6), it is convenient to use the complex variable

$$x^{(\pm)}(t) = x(t) \pm \mathrm{i}y(t) = \frac{\mathrm{sn}(t)}{(1 \mp \mathrm{i}\,\mathrm{cn}(t))}$$

and consider the complex t-plane. We take the fundamental cell for the t-plane with

$$-2K \leqslant \mathrm{Re}\, t < 2K \qquad -2K' \leqslant \mathrm{Im}\, t < 2K' \tag{21}$$

where

$$K' = \int_0^1 \frac{\mathrm{d}x}{\sqrt{(1-x^2)(1-(1-k^2)x^2)}}$$

is the complementary elliptic integral of the first kind.

Since $x^{(\pm)}(t)$ has four simple zeros in the fundamental cell at $t = -2K - 2\mathrm{i}K'$, $-2\mathrm{i}K', -2K$ and 0, the degree of the elliptic functions $x^{(\pm)}(t)$ is 4. Thus, they should have four poles in the fundamental cell. Note that

$$x^{(+)}(u + \mathrm{i}K') = \frac{1}{k\,\mathrm{sn}(u) - \mathrm{dn}(u)}.$$

From the value of k^2 in (7) and table 1, at $u = K/3$ and $5K/3, k\,\mathrm{sn}(u) = \mathrm{dn}(u)$. So, $x^{(+)}(t)$ has poles at $t = K/3 + \mathrm{i}K'$ and $5K/3 + \mathrm{i}K'$. Since $x^{(+)}(-t) = -x^{(+)}(t)$, it also has poles at $t = -(K/3 + \mathrm{i}K')$ and $-(5K/3 + \mathrm{i}K')$. We find four poles for degree 4 elliptic function $x^{(+)}(t)$, then all these poles are simple poles. Table 2 shows the poles and residues for $x^{(+)}(t) = \mathrm{sn}(t)/(1 - \mathrm{i}\,\mathrm{cn}(t))$. Here we used the following notation:

$$\alpha_1 = -K + \mathrm{i}K' \qquad \alpha_2 = K/3 + \mathrm{i}K' \qquad \alpha_3 = 5K/3 + \mathrm{i}K'. \tag{22}$$

It is clear from table 2 that all these poles are cancelled in $x^{(+)}(t) + x^{(+)}(t + 4K/3) + x^{(+)}(t - 4K/3)$. Therefore, it is constant. Evaluating this constant at $t = 0$, we get

$$x^{(+)}(t) + x^{(+)}(t + 4K/3) + x^{(+)}(t - 4K/3) = 0 \tag{23}$$

which is equivalent to (6).

Table 2. Poles and residues of $\mathrm{sn}(t)/(1-\mathrm{i}\,\mathrm{cn}(t))$ and $1/(1-\mathrm{i}\,\mathrm{cn}(t))$.

Function	$t=\alpha_2$	$t=\alpha_3$	$t=-\alpha_2$	$t=-\alpha_3$
$\mathrm{sn}(t)/(1-\mathrm{i}\,\mathrm{cn}(t))$	$\sqrt{2}/3^{1/4}$	$-\sqrt{2}/3^{1/4}$	$\sqrt{2}/3^{1/4}$	$-\sqrt{2}/3^{1/4}$
$1/(1-\mathrm{i}\,\mathrm{cn}(t))$	$1/3^{1/4}$	$-1/3^{1/4}$	$-1/3^{1/4}$	$1/3^{1/4}$

3.2. *Conservation of the moment of inertia and the angular momentum*

Let us consider a function

$$\frac{1}{1-\mathrm{i}\,\mathrm{cn}(t)}. \tag{24}$$

Since $1/(1-\mathrm{i}\,\mathrm{cn}(u+\mathrm{i}K')) = k\,\mathrm{sn}(u)/(k\,\mathrm{sn}(u)-\mathrm{dn}(u))$, function (24) has simple poles at $t=\pm\alpha_2, \pm\alpha_3$ in the above cell (21). In table 2, we list the poles and residues of this function. From this table and the value at $t=0$, it is clear that function (24) satisfies

$$\frac{1}{1-\mathrm{i}\,\mathrm{cn}(t)} + \frac{1}{1-\mathrm{i}\,\mathrm{cn}(t+4K/3)} + \frac{1}{1-\mathrm{i}\,\mathrm{cn}(t-4K/3)} = \frac{3+\sqrt{3}}{2}. \tag{25}$$

Now, consider a function $j^{(+)}(t) = x^{(-)}(t)\,\mathrm{d}x^{(+)}(t)/\mathrm{d}t$. Since $x^{(\pm)} = x \pm \mathrm{i}y$

$$j^{(+)}(t) = \left(x\frac{\mathrm{d}x}{\mathrm{d}t} + y\frac{\mathrm{d}y}{\mathrm{d}t}\right) + \mathrm{i}\left(x\frac{\mathrm{d}y}{\mathrm{d}t} - y\frac{\mathrm{d}x}{\mathrm{d}t}\right). \tag{26}$$

On the other hand, using $x^{(\pm)}(t) = \mathrm{sn}(t)/(1\mp\mathrm{i}\,\mathrm{cn}(t))$, we get

$$j^{(+)}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{1-\mathrm{i}\,\mathrm{cn}(t)} \tag{27}$$

then, from equation (25)

$$j^{(+)}(t) + j^{(+)}(t+4K/3) + j^{(+)}(t-4K/3) = 0. \tag{28}$$

Equations (26) and (28) prove the conservation of the moment of inertia and equation (9). We get equation (8) from the value at $t=0$.

3.3. *Equation of motion*

Let us consider a function

$$\triangle x^{(-)}(t) = x^{(-)}(t+4K/3) - x^{(-)}(t).$$

The function $x^{(-)}(t)$ has four simple poles at $\pm\alpha_2^*, \pm\alpha_3^*$, the star denoting complex conjugation. Then, the function $\triangle x^{(-)}(t)$ has six simple poles at $\pm\alpha_1^*, \pm\alpha_2^*, \pm\alpha_3^*$. So, the degree of this function is 6.

On $t = u + \mathrm{i}K'$, the function $\triangle x^{(-)}(t)$ is

$$\triangle x^{(-)}(u+\mathrm{i}K') = \frac{1}{k\,\mathrm{sn}(u+4K/3)+\mathrm{dn}(u+4K/3)} - \frac{1}{k\,\mathrm{sn}(u)+\mathrm{dn}(u)}.$$

And from table 1, we see $\mathrm{sn}(5K/3) = \mathrm{sn}(K/3)$ and $\mathrm{dn}(5K/3) = \mathrm{dn}(K/3)$. Thus, at $u = K/3$, i.e., $t = K/3 + \mathrm{i}K'$, $\triangle x^{(-)}(t)$ has a zero point. To see the behaviour of the function around the zero point, let us take $u = K/3 + \triangle u$. Then, using $\mathrm{sn}(2K-z) = \mathrm{sn}(z)$ and $\mathrm{dn}(2K-z) = \mathrm{dn}(z)$, we get

$$\triangle x^{(-)}(K/3+\triangle u+\mathrm{i}K') = -\triangle x^{(-)}(K/3-\triangle u+\mathrm{i}K')$$

i.e., $\triangle x^{(-)}(t)$ is an odd function around the zero point $t = \alpha_2 = K/3 + \mathrm{i}K'$.

Table 3. Principal part around the pole with $a = 2\sqrt{2}/3^{1/4}$, $b = 3^{1/4}/\sqrt{2}$.

Function	$t = \alpha_2$	$t = \alpha_3$	$t = -\alpha_2$	$t = -\alpha_3$
$[\triangle x^{(-)}(t)]^{-1}$	$\frac{2a}{(t-\alpha_2)^3} - \frac{b}{t-\alpha_2}$			$-\frac{2a}{(t+\alpha_3)^3} + \frac{b}{t+\alpha_3}$
$-\left[\triangle x^{(-)}\left(t - \frac{4K}{3}\right)\right]^{-1}$		$-\frac{2a}{(t-\alpha_3)^3} + \frac{b}{t-\alpha_3}$	$\frac{2a}{(t+\alpha_2)^3} - \frac{b}{t+\alpha_2}$	
$\mathrm{d}^2x^{(+)}(t)/\mathrm{d}t^2$	$\frac{a}{(t-\alpha_2)^3}$	$-\frac{a}{(t-\alpha_3)^3}$	$\frac{a}{(t+\alpha_2)^3}$	$-\frac{a}{(t+\alpha_3)^3}$
$x^{(+)}(t)$	$\frac{1}{b(t-\alpha_2)}$	$-\frac{1}{b(t-\alpha_3)}$	$\frac{1}{b(t+\alpha_2)}$	$-\frac{1}{b(t+\alpha_3)}$

Series expansion of $x^{(-)}(u + 4K/3 + \mathrm{i}K')$ and $x^{(-)}(u + \mathrm{i}K')$ around $u = K/3$ yields

$$x^{(-)}(u + 4K/3 + \mathrm{i}K') = \frac{1}{\sqrt{2}} + \frac{1}{8}\sqrt{\frac{3}{2}}(u - K/3)^2 + \frac{3^{1/4}}{8\sqrt{2}}(u - K/3)^3 + \frac{9}{128\sqrt{2}}(u - K/3)^4 + \frac{3^{3/4}}{64\sqrt{2}}(u - K/3)^5 + \cdots \tag{29}$$

$$x^{(-)}(u + \mathrm{i}K') = \frac{1}{\sqrt{2}} + \frac{1}{8}\sqrt{\frac{3}{2}}(u - K/3)^2 - \frac{3^{1/4}}{8\sqrt{2}}(u - K/3)^3 + \frac{9}{128\sqrt{2}}(u - K/3)^4 - \frac{3^{3/4}}{64\sqrt{2}}(u - K/3)^5 + \cdots. \tag{30}$$

Thus, we get

$$\triangle x^{(-)}(t) = \frac{3^{1/4}}{4\sqrt{2}}(t - \alpha_2)^3 + \frac{3^{3/4}}{32\sqrt{2}}(t - \alpha_2)^5 + \cdots \tag{31}$$

and

$$\frac{1}{\triangle x^{(-)}(t)} = \frac{4\sqrt{2}}{3^{1/4}}\frac{1}{(t - \alpha_2)^3} - \frac{3^{1/4}}{\sqrt{2}}\frac{1}{(t - \alpha_2)} + \cdots \tag{32}$$

i.e., $1/\triangle x^{(-)}(t)$ has a triple pole at $t = \alpha_2$. Since $x^{(-)}(-t) = -x^{(-)}(t)$, $\triangle x^{(-)}(t)$ also has a triple zero at $t = -\alpha_3 = -5K/3 - \mathrm{i}K'$, as follows:

$$\triangle x^{(-)}(-\alpha_3 + \triangle t) = \triangle x^{(-)}(\alpha_2 - \triangle t).$$

Thus, $1/\triangle x^{(-)}(t)$ has a triple pole at $t = -\alpha_3$ with the principal part

$$\frac{1}{\triangle x^{(-)}(t)} = -\frac{4\sqrt{2}}{3^{1/4}}\frac{1}{(t + \alpha_3)^3} + \frac{3^{1/4}}{\sqrt{2}}\frac{1}{(t + \alpha_3)} + \cdots. \tag{33}$$

We found two triple poles for the elliptic function of degree 6.

In table 3, we list the principal part of $1/\triangle x^{(-)}(t)$, $-1/\triangle x^{(-)}(t - 4K/3)$, $\mathrm{d}^2x^{(+)}(t)/\mathrm{d}t^2$ and $x^{(+)}(t)$. From this table, we see the following equation is satisfied,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x^{(+)}(t) = \frac{1}{2}\left\{\frac{1}{\triangle x^{(-)}(t)} - \frac{1}{\triangle x^{(-)}\left(t - \frac{4K}{3}\right)}\right\} + \frac{\sqrt{3}}{4}x^{(+)}(t). \tag{34}$$

This is the equation of motion in (13) with the repulsive force in (14).

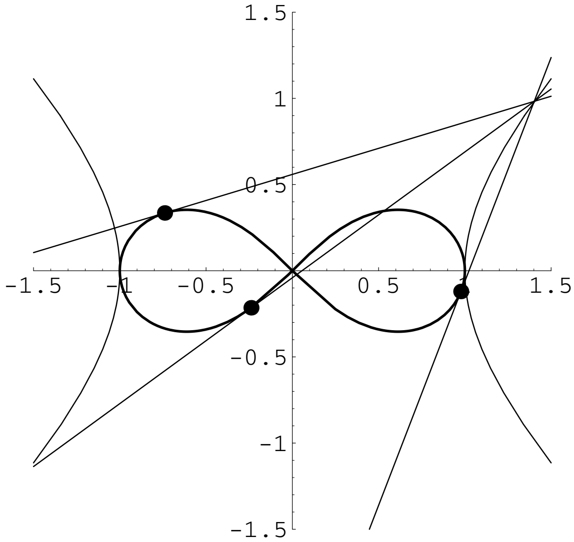

Figure 2. Snapshot at $t = -K/6$. Three tangent lines from the choreographic three bodies on the lemniscate meet at a point on the rectangular hyperbola. Full circles ● are the positions of the three bodies.

4. Choreographic three bodies on the lemniscate and the rectangular hyperbola

In the general three-body problem on the plane, not restricted to this lemniscate case, the conservation of the centre of mass and zero angular momentum has a simple geometrical meaning: three tangent lines from the three bodies must meet at a point (infinity is allowed) at each instant. That is, there exist scalars $\lambda_i(t)$ and a vector $\mathbf{c}(t)$, such that

$$\mathbf{c} = \mathbf{x}_i + \lambda_i \mathbf{v}_i \qquad \text{for} \quad i = 1, 2, 3. \tag{35}$$

The solution λ_i is

$$\lambda_i = \frac{\{(\mathbf{x}_j - \mathbf{x}_i) \times \mathbf{v}_j\} \cdot \hat{\mathbf{z}}}{(\mathbf{v}_i \times \mathbf{v}_j) \cdot \hat{\mathbf{z}}} \qquad \text{with} \quad \forall j \neq i \tag{36}$$

and the vector $\mathbf{c}(t)$ is given by

$$\mathbf{c} = -\frac{l_i \mathbf{v}_j - l_j \mathbf{v}_i}{(\mathbf{v}_i \times \mathbf{v}_j) \cdot \hat{\mathbf{z}}} \qquad \text{with } (i, j) = (1, 2), (2, 3), (3, 1) \tag{37}$$

where $\hat{\mathbf{z}} = \hat{\mathbf{x}} \times \hat{\mathbf{y}}$ and $l_i = (\mathbf{x}_i \times \mathbf{v}_i) \cdot \hat{\mathbf{z}}$.

For the lemniscate, we find that the orbit of $\mathbf{c}(t)$ is the rectangular hyperbola as shown in figure 2,

$$(\mathbf{c} \cdot \hat{\mathbf{x}})^2 - (\mathbf{c} \cdot \hat{\mathbf{y}})^2 = 1 \tag{38}$$

by a direct algebraic calculation.

Let us introduce the terms 'forward' and 'backward' for the direction of the choreographic motion on the lemniscate. We call the motion 'forward' if points pass through the origin upward (from lower left to upper right or from lower right to upper left) and call it 'backward' otherwise. In other words, a 'forward' motion is a clockwise motion on the right leaf of the lemniscate or an anticlockwise motion on the left leaf.

We observed the following two properties: (i) for the 'forward' motion of the choreographic three bodies, the point $\mathbf{c}$ always moves upward on the rectangular hyperbola.

(ii) The four positions of the choreographic three bodies and the cross point on the rectangular hyperbola, $\mathbf{x}_i, i = 1, 2, 3$ and $\mathbf{c}$, are in different quadrants as shown in figure 2. Note that when the point $\mathbf{c}$ jumps the leaf of the hyperbola one of the three bodies passes through the origin. And when the point $\mathbf{c}$ passes through the horizontal axis upward (downward) one of the three bodies passes through the horizontal axis at the same point in the opposite direction, downward (upward).

Then, we will have two questions as to how the choreographic three points can be determined geometrically. The first question: How can we find the positions of the choreographic three bodies for an arbitrarily given point $\mathbf{c}$ on the rectangular hyperbola? We can draw four (in general) tangent lines to the lemniscate from the point $\mathbf{c}$. Therefore, we have four (in general) contact points instead of three. How can we select the choreographic three points from among them? We observed that the following two methods work: (i) move the point $\mathbf{c}$ slightly upward. Then, we will observe that three of the four contact points move 'forward', and one 'backward'. Of course, the former are the positions of the choreographic three bodies. (ii) Choose three points which are not in the same quadrants the given point $\mathbf{c}$ is in.

The second question: How can we find the other two choreographic positions for an arbitrarily given point $\mathbf{x}_1$ on the lemniscate? The tangent lines of the lemniscate which contact at the given point $\mathbf{x}_1$ have two cross points, $\mathbf{d}_1, \mathbf{d}_2$, on the rectangular hyperbola. How can we select one? Again we have two methods. (i) If we move the point $\mathbf{x}_1$ 'forward', one of $\mathbf{d}_i$ will move upward, and the other downward. The former is the point $\mathbf{c}$ the point $\mathbf{x}_1$ corresponds to. (ii) Select a cross point $\mathbf{d}_i$ which is in a different quadrant from the one the given point $\mathbf{x}_1$ is in. Using the method described in the above paragraph, we can find positions $\mathbf{x}_2$ and $\mathbf{x}_3$ for the other two choreographic bodies.

The above procedures give a geometrical method to find a set of positions of the choreographic three bodies on the lemniscate.

5. Summary and discussions

We have shown that the choreographic three bodies on the lemniscate (1), (3) and (7) satisfy the equation of motion under two different potential energies, U defined in (16) and V in (17). It means that the choreographic three bodies on the lemniscate can be realized in the following two ways: (i) consider four particles, the infinitely heavy particle 0 and three particles 1–3 of equal mass. Take the universal gravitation in two dimensions

$$U_{ij} = \frac{1}{2}\ln r_{ij} \tag{39}$$

as the interaction potential energy between particles i and j with $i, j = 1, 2, 3$, and the artificial repulsive potential energy

$$U_{0i} = -\frac{\sqrt{3}}{8}\mathbf{x}_i^2 \tag{40}$$

as the interaction potential energy between particles 0 and i with $i = 1, 2, 3$. Then, put the infinitely heavy particle 0 at the origin and put the three particles 1–3 on the lemniscate according to equation (3). (ii) Consider three particles 1–3 of equal mass. Take the universal gravitation in two dimensions accompanied by the artificial repulsive potential

$$V_{ij} = \frac{1}{2}\ln r_{ij} - \frac{\sqrt{3}}{24}r_{ij}^2 \tag{41}$$

as the interaction potential energy between particles i and j with $i, j = 1, 2, 3$. Then, put the three particles 1–3 on the lemniscate according to equation (3).

As a consequence of the equation of motion, the choreographic three bodies on the lemniscate satisfy the kinematical conservation laws independent of the potential energies, i.e., the centre of mass (6) and the angular momentum (9). Furthermore, they conserve a lot of kinematical and geometrical quantities, the kinetic energy (11), the moment of inertia (8), the product of mutual distances (18), the sum of squares of mutual distances (12) and the sum of squares of the curvature (10).

We have also pointed out that the tangent lines from the positions of the choreographic three bodies on the lemniscate meet at a point on the rectangular hyperbola, then we have given the two geometrical methods to determine the choreographic three points on the lemniscate. We have given: (i) how to determine the choreographic three bodies from a point on the rectangular hyperbola and (ii) how to determine the choreographic three points from the given first point on the lemniscate.

For Moore, Chenciner, Montgomery and Simó's figure-eight choreography (Simó's E orbit), no conservation laws are known besides the centre of mass, total energy and angular momentum. We expect that all properties described in section 4 are also valid for their choreography. But the algebraic or transcendent character of their figure-eight orbit and orbit of cross point of three tangent lines still remains unknown. Especially, it is not clear how many tangent lines can be drawn to their figure-eight from a given cross point. For Simó's non-choreographic H1, H2 and H3 orbits [4], the situation is more complicated. Further investigations are needed for these orbits.

The affine transformed motion $x(t)\hat{\mathbf{x}} + k^2 y(t)\hat{\mathbf{y}}$ of lemniscate (1) with the modulus (7) is numerically close to Simó's figure-eight choreography (E orbit). It is difficult to distinguish the two orbits if they are plotted on the usual computer display. We believed for some time that the choreographic three bodies on the lemniscate was Simó's figure-eight choreography (E orbit) because of the numerical similarity and a lot of conservation quantities of them. However, any affine transformation of the lemniscate cannot be Simó's figure-eight orbit as he pointed out [4]. We hope that we can discuss in the future the orbit of Moore, Chenciner, Montgomery and Simó's figure-eight motion in our context.

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References

[1] Moore C 1993 *Phys. Rev. Lett.* **70** 3675–9

[2] Chenciner A and Montgomery R 2000 *Ann. Math.* **152** 881–901

[3] Simó C 2001 *Proc. 3rd European Cong. Math. (Progress in Mathematics vol 201)* (Basel: Birkhauser) pp 101–15

[4] Simó C 2002 *Celestial Mechanics: Dedicated to Donald Saari for His 60th Birthday (Contemporary Mathematics vol 292)* (Providence, RI: American Mathematical Society) pp 209–28

[5] Spiegel M R 1967 *Theory and Problems of Theoretical Mechanics* (New York: McGraw-Hill) p 138